FURTHER MATHS

AS FURTHER MATHS - Chapter 1 - **Complex numbers** - CYCLE 1

1 $\mathbf{i = \sqrt{-1}}$ and $\mathbf{i^2 = -1}$

2 An **imaginary number** is a number of the form $b\text{i}$, where $b \in \mathbb{R}$.

3 A **complex number** is written in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

4 Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.

5 You can multiply a real number by a complex number by multiplying out the brackets in the usual way.

6 If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which is real.

7 For any complex number $z = a + b\text{i}$, the **complex conjugate** of the number is defined as $z^* = a - b\text{i}$.

8 For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.

9 If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.

10 If $\text{f}(z)$ is a polynomial with real coefficients, and z_1 is a root of $\text{f}(z) = 0$, then $z_1{}^*$ is also a root of $\text{f}(z) = 0$.

11 An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.

For a cubic equation with real coefficients, either:

- all three roots are real, or
- one root is real and the other two roots form a complex conjugate pair.

12 An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.

For a quartic equation with real coefficients, either:

- all four roots are real, or
- two roots are real and the other two roots form a complex conjugate pair, or
- two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

FURTHER MATHS

AS FURTHER MATHS - Chapter 2 - **Argand diagrams** - CYCLE 1

1 You can represent complex numbers on an **Argand diagram**. The x-axis on an Argand diagram is called the **real axis** and the y-axis is called the **imaginary axis**. The complex number $z = x + iy$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.

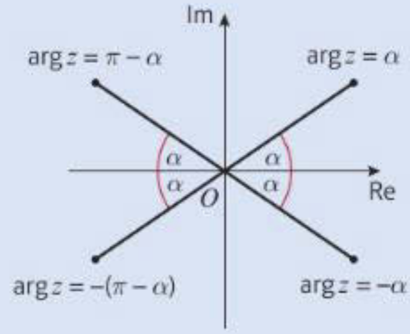
2 The complex number $z = x + iy$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.

3 The **modulus** of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.

4 The **argument** of a complex number, $\arg z$, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram. For a complex number $z = x + iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$

5 Let α be the positive acute angle made with the real axis by the line joining the origin and z.

- If z lies in the first quadrant then $\arg z = \alpha$.
- If z lies in the second quadrant then $\arg z = \pi - \alpha$.
- If z lies in the third quadrant then $\arg z = -(\pi - \alpha)$.
- If z lies in the fourth quadrant then $\arg z = -\alpha$.



6 For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + i\sin\theta)$

AS FURTHER MATHS - Chapter 3 - **Series** - CYCLE 1

1 To find the sum of a series of constant terms you can use the formula $\sum_{r=1}^{n} 1 = n$.

2 The formula for the sum of the first n natural numbers is $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

3 To find the sum of a series that does not start at $r = 1$, use $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$

4 You can rearrange expressions involving sigma notation.

- $\sum_{r=1}^{n} kf(r) = k\sum_{r=1}^{n} f(r)$
- $\sum_{r=1}^{n} (f(r) + g(r)) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

5 The formula for the sum of the squares of the first n natural numbers is

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

6 The formula for the sum of the cubes of the first n natural numbers is

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

FURTHER MATHS

AS FURTHER MATHS - Chapter 4 - **Roots of polynomials** - CYCLE 2

1 If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2 If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:

- $\alpha + \beta + \gamma = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma = -\frac{d}{a}$

3 If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:

- $\alpha + \beta + \gamma + \delta = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \Sigma\alpha\beta\gamma = -\frac{d}{a}$
- $\alpha\beta\gamma\delta = \frac{e}{a}$

AS FURTHER MATHS - Chapter 6 - **Matrices** - CYCLE 2

1 A **square matrix** is one where the numbers of rows and columns are the same.

2 A zero matrix is one in which all of the numbers are zero. The zero matrix is denoted by 0.

3 An identity matrix is a square matrix in which the numbers in the leading diagonal (starting top left) are 1 and all the rest are 0. Identity matrices are denoted by $\mathbf{I}_k$ where k describes the size. The 3 × 3 identity matrix is $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4 To add or subtract matrices, you add or subtract the corresponding elements in each matrix. You can only add or subtract matrices that are the same size.

5 To multiply a matrix by a scalar, you multiply every element in the matrix by that scalar.

6
- Matrices can be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix. In this case the first is said to be multiplicatively conformable with the second.
- To find the product of two multiplicatively conformable matrices, you multiply the elements in each row in the left-hand matrix by the corresponding elements in each column in the right-hand matrix, then add the results together.

7 For a 2 × 2 matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant of $\mathbf{M}$ is $ad - bc$.

8
- If det $\mathbf{M} = 0$ then $\mathbf{M}$ is a **singular** matrix.
- If det $\mathbf{M} \neq 0$ then $\mathbf{M}$ is a **non-singular** matrix.

FURTHER MATHS

9 You find the determinant of a 3 × 3 matrix by reducing the 3 × 3 determinant to 2 × 2 determinants using the formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

10 The **minor** of an element in a 3 × 3 matrix is the determinant of the 2 × 2 matrix that remains after the row and column containing that element have been crossed out.

11 The **inverse** of a matrix **M** is the matrix $\mathbf{M}^{-1}$ such that $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

12 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

13 If **A** and **B** are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

14 The **transpose** of a matrix is found by interchanging the rows and the columns.

15 Finding the inverse of a 3 × 3 matrix **A** usually consists of the following 5 steps.

Step 1 Find the determinant of **A**, det **A**.

Step 2 Form the matrix of the minors of **A**, **M**.

In forming the matrix **M**, each of the nine elements of the matrix **A** is replaced by its minor.

Step 3 From the matrix of minors, form the matrix of **cofactors**, **C**, by changing the signs of some elements of the matrix of minors according to the **rule of alternating signs** illustrated by the pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

You leave the elements of the matrix of minors corresponding to the + signs in this pattern unchanged. You change the signs of the elements corresponding to the – signs.

Step 4 Write down the transpose, $\mathbf{C}^{T}$, of the matrix of cofactors.

Step 5 The inverse of the matrix **A** is given by the formula

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\mathbf{C}^{T}$$

16 If $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{v}$ then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\mathbf{v}$.

17 A system of linear equations is **consistent** if there is at least one set of values that satisfies all the equations simultaneously. Otherwise, it is **inconsistent**.

AS FURTHER MATHS - Chapter 7 - **Linear Transformations** - CYCLE 2

1
- Linear transformations always map the origin onto itself.
- Any linear transformation can be represented by a matrix.

2 The linear transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ can be represented by the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$.

3 A reflection in the y-axis is represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Points on the y-axis are invariant points, and the lines $x = 0$ and $y = k$ for any value of k are invariant lines.

4 A reflection in the x-axis is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Points on the x-axis are invariant points, and the lines $y = 0$ and $x = k$ for any value of k are invariant lines.

5 A reflection in the line $y = x$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Points on the line $y = x$ are invariant points, and the lines $y = x$ and $y = -x + k$ for any value of k are invariant lines.

6 A reflection in the line $y = -x$ is represented by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Points on the line $y = -x$ are invariant points, and the lines $y = -x$ and $y = x + k$ for any value of k are invariant lines.

AS FURTHER MATHS - Chapter 5 - **Volumes of revolution** - CYCLE 2

1 The volume of revolution formed when $y = \mathrm{f}(x)$ is rotated about the x-axis between $x = a$ and $x = b$ is given by

$$\textbf{Volume} = \pi \int_a^b y^2 \, \mathrm{d}x$$

2 The volume of revolution formed when $x = \mathrm{f}(y)$ is rotated about the y-axis between $y = a$ and $y = b$ is given by

$$\textbf{Volume} = \pi \int_a^b x^2 \, \mathrm{d}y$$

3 A cylinder of height h and radius r has volume $\pi r^2 h$.

4 A cone of height h and base radius r has volume $\frac{1}{3}\pi r^2 h$.

FURTHER MATHS

FURTHER MATHS

AS FURTHER MATHS - Chapter 9 - **Vectors** - CYCLE 3

1 A vector equation of a straight line passing through the point A with position vector **a**, and parallel to the vector **b**, is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a scalar parameter.

2 A vector equation of a straight line passing through the points C and D, with position vectors **c** and **d** respectively, is

$$\mathbf{r} = \mathbf{c} + \lambda(\mathbf{d} - \mathbf{c})$$

where λ is a scalar parameter.

3 If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the equation of the line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ can be given in Cartesian form as:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

Each of these three expressions is equal to λ.

4 The vector equation of a plane is

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where:

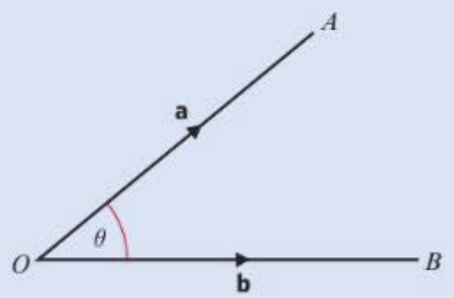
- **r** is the position vector of a general point in the plane
- **a** is the position vector of a point in the plane
- **b** and **c** are non-parallel, non-zero vectors in the plane
- λ and μ are scalars

5 A Cartesian equation of a plane in three dimensions can be written in the form $ax + by + cz = d$ where a, b, c and d are constants, and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the normal vector to the plane.

6 The **scalar product** of two vectors **a** and **b** is written as **a.b** (say '**a** dot **b**'), and defined as

$$\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between **a** and **b**.



7 If **a** and **b** are the position vectors of the points A and B, then $\cos(\angle AOB) = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$

8 The non-zero vectors **a** and **b** are perpendicular if and only if $\mathbf{a.b} = 0$.

9 If **a** and **b** are parallel, $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|$. In particular, $\mathbf{a.a} = |\mathbf{a}|^2$.

10 If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$$\mathbf{a.b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

11 The acute angle θ between two intersecting straight lines is given by

$$\cos\theta = \left|\frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}\right|$$

where **a** and **b** are direction vectors of the lines.

12 The scalar product form of the equation of a plane is $\mathbf{r.n} = k$ where $k = \mathbf{a.n}$ for any point in the plane with position vector **a**.

13 The acute angle θ between the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation $\mathbf{r.n} = k$ is given by the formula

$$\sin\theta = \left|\frac{\mathbf{b.n}}{|\mathbf{b}||\mathbf{n}|}\right|$$

14 The acute angle θ between the plane with equation $\mathbf{r.n}_1 = k_1$ and the plane with equation $\mathbf{r.n}_2 = k_2$ is given by the formula

$$\cos\theta = \left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|$$

AS FURTHER MATHS - Chapter 8 - **Proof by induction** - CYCLE 3

1 You can use **proof by induction** to prove that a general statement is true for all positive integers.

2 Proof by mathematical induction usually consists of the following four steps:

- **Basis:** Show the general statement is true for $n = 1$.
- **Assumption:** Assume that the general statement is true for $n = k$.
- **Inductive:** Show the general statement is true for $n = k + 1$.
- **Conclusion:** State that the general statement is then true for all positive integers, n.

FURTHER MATHS